Second-order cone optimization for unilateral contact with Coulomb friction

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CMIS 2009
Chania, Apr. 28th

Plan

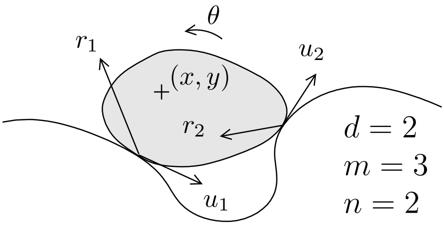
1. Model
2. Painlevé-Klein example
3. SOCP reformulation
4. Theoretical interest
5. Numerical interest

Plan

Mechanical problem

Consider a mechanical system in dimension d ($d = 2$ or 3) with

- finitely many degrees of freedom: m
- finitely many contact points: n
- Coulomb friction at contact points
- inelastic impacts for simplicity



Notations

Problem: compute one time step

Unknowns:

- $v \in \mathbb{R}^m$: (discretized) generalized velocities
- $u \in \mathbb{R}^{nd}$: (discretized) relative velocities at contact points
- $r \in \mathbb{R}^{nd}$: (discretized) contact forces or impulses

Previous example:

- $v \approx (\dot{x}, \dot{y}, \dot{\theta}) \in \mathbb{R}^3$
- $u \approx (u^1, u^2) \in \mathbb{R}^4$
- $r \approx (r^1, r^2) \in \mathbb{R}^4$

Assumptions

linear dynamics

$$Mv + f = H^\top r \tag{1}$$

linear kinematics

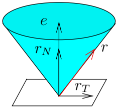
$$u = Hv + w \tag{2}$$

for given

- M (size: $m \times m$, symmetric positive)
- f (size: $m \times 1$)
- H (size: $nd \times m$)
- w (size: $nd \times 1$)

Assumptions

Given normal vector e, define second-order cone K_μ



$$K_\mu = \{\|r_T\| \leq \mu r_N\} \subset \mathbb{R}^3$$

Coulomb friction (at velocity level)

$$\begin{cases} \text{or: take-off} & r = 0 \text{ and } u_N \geq 0 \\ \text{or: stick} & r \in int(K_\mu) \text{ and } u = 0 \\ \text{or: slide} & r \in \partial K_\mu \setminus 0 \text{ and } u_N = 0 \\ & \text{with } u_T \text{ opposed to } r_T \end{cases} \quad (3)$$

We note (3) by

$$(u, r) \in C(\mu, e)$$

Known reformulations of $C(\mu, e)$

Coulomb's law (3) can be equivalently reformulated

Alart-Curnier

non-linear, non-smooth equation $f_{AC}(u, r) = r$ with

$$f_{AC}(u, r) = \begin{bmatrix} \mathrm{P}_{\mathbb{R}^+}(r_N - \rho u_N) \\ \mathrm{P}_{B(0, \mu r_N)}(r_T - \rho u_T) \end{bmatrix}$$

De Saxcé

complementarity constraint $K_\mu^* \ni \widetilde{u} \perp r \in K_\mu$ with

$$\widetilde{u} := u + \mu \|u_T\| e$$

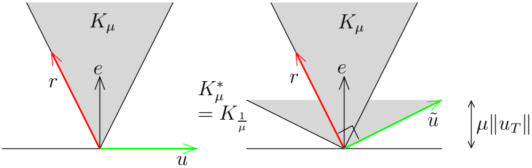
K^*: dual cone of K; $K_\mu^* = K_{\frac{1}{\mu}}$

Interpretation of $\widetilde{u}$

The change of variables

$$\widetilde{u} := u + \mu \| u_T \| e$$

introduces normality



but the value of $\| u_T \|$ is unknown a priori !

Whole problem

- Finally, we want to solve the incremental problem

$$\left\{ \begin{array}{lcl} Mv + f & = & H^\top r \\ u & = & Hv + w \\ K_\mu^* \ni \widetilde{u} & \perp & r \in K_\mu \end{array} \right. \tag{4}$$

- algorithms are available
  - Newton iterations on Alart-Curnier function
  - Uzawa iterations on De Saxcé's bipotential
  - many variants (fixed-points iterations, "Gauss-Seidel" ...)
- ...but algorithms sometimes fail
  - does a solution exist ?
  - if "yes", improve our algorithm !
  - if "no", does our problem make sense ?

Plan

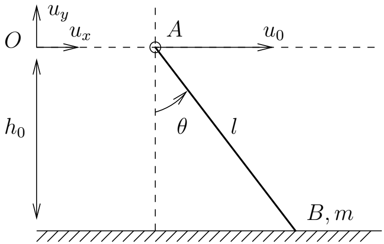
Example

Consider a rigid bar under gravity with $d = 2$ and

- imposed velocity u_0 in A
- one degree of freedom θ
- $v = 0$ at previous time step

Constants:

- do not fix yet $u_0 \neq 0$, μ and h_0 (or $\theta \in]0, \frac{\pi}{2}[$)
- other values fixed for convenience



Incremental problem

After time discretization

$$\left\{\begin{array}{rcl} v & = & \cos(\theta) r_x + \sin(\theta) r_y - \sin\theta \\ u_x & = & \cos(\theta) v + u_0 \\ u_y & = & \sin(\theta) v \\ (u, r) & \in & C(u_y, \mu) \end{array}\right.$$

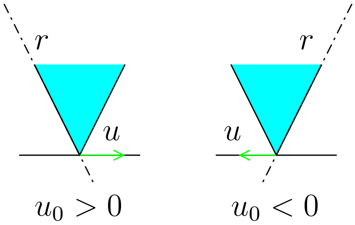
which can be solved by inspection

- one contact $\longrightarrow$ only three cases
- $d = 2 \longrightarrow$ all constraints are linear

Solving by hand

Consider the three possible cases

- take-off: $r = 0$ implies $u_y < 0$, impossible
- stick: $u = 0$. If $u_0 \neq 0$, impossible
- slide: $u_y = 0$ implies $v = 0$ and $u_x = u_0$ (to be continued. . .)



Case $u_0 < 0$

If $u_0 < 0$,

- linear system (plus condition $r_y \geq 0$)

$$\left\{ \begin{array}{lcl} \cos(\theta) r_x + \sin(\theta) r_y & = & \sin(\theta) \\ -r_x + \mu r_y & = & 0 \end{array} \right.$$

- solution is

$$r_y = \frac{\tan(\theta)}{\tan(\theta) + \mu} \geq 0$$

- condition $r_y \geq 0$ is automatically satisfied

A (unique) solution exists

Case $u_0 > 0$

If $u_0 > 0$,

- linear system (plus condition $r_y \geq 0$)

$$\left\{ \begin{array}{lcl} \cos(\theta) r_x + \sin(\theta) r_y & = & \sin(\theta) \\ r_x + \mu r_y & = & 0 \end{array} \right.$$

- solution is

$$r_y = \frac{\tan(\theta)}{\tan(\theta) - \mu}$$

for $\tan\theta \neq \mu$ (otherwise, no solution)

- condition $r_y \geq 0$ gives:

$$\tan\theta > \mu$$

A solution exists if and only if $\tan\theta > \mu$

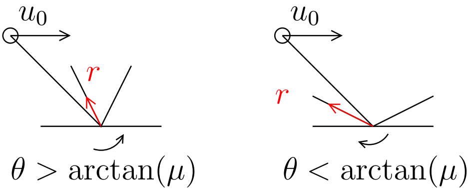
Conclusion

Finally. . .

A solution exists iff $u_0 \leq 0$ or $[u_0 > 0$ and $\tan(\theta) > \mu]$

Coherent with intuition:

- when $\tan\theta > \mu$, friction torque acts counter-clockwise,
- compensate effect of gravity
- if $\tan\theta < \mu$, friction torque acts clockwise,
- and increases effect of gravity



Goal

- Given the data M, f, H, w (plus μ and e)
- provide a checkable criterion
- that ensures existence of a solution

Validation:
the criterion can be checked on the Painlevé-Klein example

Plan

Reformulation

- Introduce extra variable s^i at each contact

$$s^i := \|u_T^i\| \tag{5}$$

- perform the change of variables (cf De Saxcé)

$$u \longrightarrow \widetilde{u} := u + \mu e s$$

incremental problem

$$\left\{ \begin{array}{lcl} Mv + f & = & H^\top r \\ \widetilde{u} & = & Hv + w + \mu e s \\ K_\mu^* \ni \widetilde{u} \perp r \in K_\mu \end{array} \right. \tag{6}$$

Why should we do that ?

(6) are KKT conditions of two convex optimization problems (SOCP: second order cone programs)

primal problem

$$\begin{cases} \min \quad J(v) := \frac{1}{2} v^\top M v + f^\top v \\ Hv + w + \alpha s \in K_\mu^* \end{cases} \qquad (D_s)$$

dual problem

$$\begin{cases} \min \quad J_s(r) := \frac{1}{2} r^\top W r - b_s^\top r \\ r \in K_\mu \end{cases} \qquad (P_s)$$

with $W = HM^{-1}H^\top$ and $b_s = \alpha s + \beta$

Side note: when $\mu = 0$, incremental problem is a QP (Moreau)

Final reformulation

- Introducing

$$u(s) := \mathrm{argmin}_u(P_s) = \mathrm{argmin}_u(D_s)$$

practically computable by optimization software, and

$$F^i(s) := \|u^i_T(s)\|,$$

- the incremental problem becomes

fixed point problem

$$F(s) = s$$

Plan

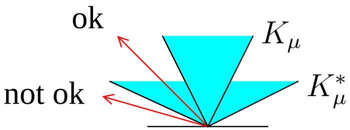
Assumption

$$\exists v \in \mathbb{R}^m \;:\; Hv + w \in K_\mu^*$$

- Interpretation: it is kinematically possible to enforce

$$u \in K_\mu^*$$

  at each contact
- not only the intuitive $u_N \geq 0$!



Consequence

Using the assumption,

- the application $F : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is well-defined
- it is continuous
- it is bounded
- apply Brouwer's theorem

Theorem

A fixed point exists

Application to the Painlevé example

For the example

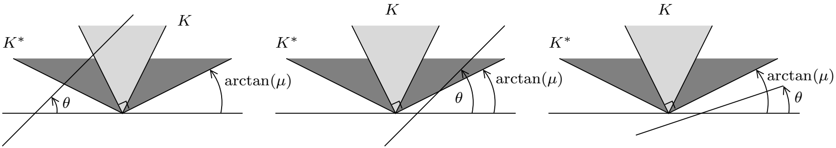
- the criterion is

$$\exists v \in \mathbb{R} \; ; \; (u_0, 0) + (\cos\theta, \sin\theta)\, v \in K_\mu^*$$

- we find exactly

condition

$$u_0 \leq 0 \text{ or } [u_0 > 0 \text{ and } \tan(\theta) > \mu]$$



$u_0 < 0$ $\quad$ $u_0 > 0$ and $\theta > \arctan(\mu)$ $\quad$ $u_0 > 0$ and $\theta \leq \arctan(\mu)$

Applicability of the criterion

- determine whether the intersection of a cone and a affine halfspace is empty of not
- easily checkable in general
- only a sufficient condition
- necessary and sufficient for Painlevé-Klein example . . .
- . . . and when when $\mu = 0$ (QP case)
- not true in general

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Can this be used in practice ?

The fixed point equation $F(s) = s$ can be tackled by

- fixed-point iterations

$$s \leftarrow F(s)$$

- Newton iterations

$$s \leftarrow \mathrm{Jac}[F](s) \backslash F(s)$$

- Variants possible (truncated resolution of inner problem...)

Does it work ?

- fixed-point iterations:
  - expensive
  - not very robust
- Newton:
  - usually very few iterations...
  - ...but they are expensive
- bottleneck: SOCP solver
- practical interest is unclear yet
  - more robust ?
  - faster ?

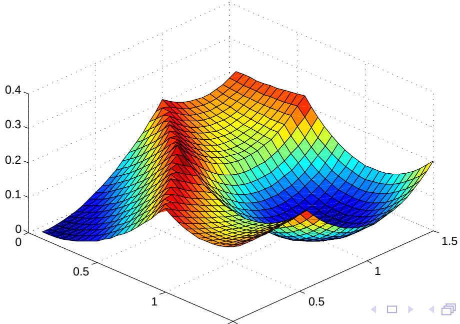
Illustration

- Toy problem: $d = 2$, $n = 2$ (a bead stuck in a corner)
- plotting

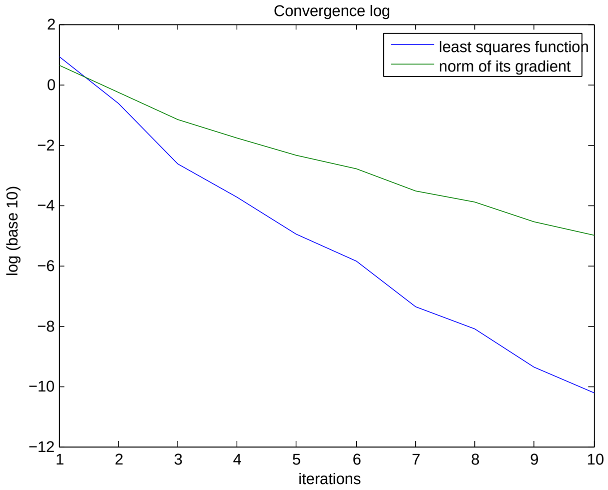
$$s \longrightarrow \frac{1}{2}\|F(s) - s\|^2$$

observe

- non-smoothness...
- ...and non-convexity
- ...and non-uniqueness



Convergence log (3D, 100 contacts)



Summary

- a new formulation of (standard) incremental problem
- yields a checkable existence criterion
- and new numerical prospects

Side notes:

- anisotropic friction could be handled
- non-linear dynamics as well

Thanks for your attention !

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